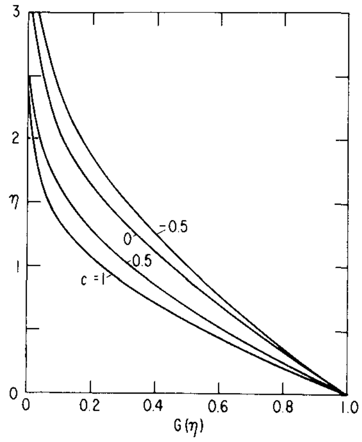

Fig. 2 The normalized y-wise velocity profiles for various values of the parameter c.

$$
\begin{aligned}
f''' + (f+cg)f'' + (1-f'^2) &= 0 \\
g''' + (f+cg)g'' + c(1-g'^2) &= 0
\end{aligned}
\tag{6}
$$

The appropriate boundary conditions are

$$\text{at } \eta = 0: \quad f = g = f' = g' = 0, \quad F = G = 1$$

$$\text{at } \eta \to \infty: \quad f' = g' = 1, \quad F = G = 0$$

Equations (6) depend only on the parameter c and have been solved numerically by Davey[5] for a range of values of that parameter. Rott[2] provides for the case $c = 0$ the solution to Equations (5); in particular he finds $F'(0) = -0.811$. We give in Table 1 the crucial wall values, $F'(0)$ and $G'(0)$, for several values of c, and show in Figs. 1 and 2 the corresponding profiles, $F(\eta)$ and $G(\eta)$.‡ These results have been obtained by straightforward numerical means.

The values $F(0)$, $G(0)$ permit the description of the wall shear. Consider a line $y_{ws}(x)$ such that the wall shear is everywhere tangent thereto. Then y_{ws} corresponds to a solution of the equation

$$\frac{dy_{ws}}{dx} = \frac{\tau_y}{\tau_x} = \frac{v_w G'(0) + c\alpha y_{ws} g''(0)}{u_w F'(0) + \alpha x f''(0)} \tag{7}$$

Let $Y \equiv \alpha y_{ws}/u_w$, $\xi \equiv \alpha x/u_w$, $\alpha \equiv -F'(0)/cg''(0)$, $\beta \equiv f''(0)/cg''(0)$, and $\gamma \equiv -v_w G'(0)/u_w g''(0)$. The solution of Eq. (7) is readily found to be

$$Y - \gamma = C(\beta\xi - \alpha)^{1/\beta} \tag{8}$$

where C is a constant of integration which may be conveniently evaluated by specifying a particular line on the surface, e.g., the value of Y at $\xi = 0$. The line $\beta\xi - \alpha = 0$ is clearly important.

Consider, for clarity, a special case in which $c > 0$ and the surface is moving in only the x-direction so that $\gamma = 0$ and $u_w > 0$. For $\xi < 0$ there is "reverse" flow in that $u > 0$ for some region close to the wall and $u < 0$ beyond that region. The aforementioned line corresponding to $\xi = \alpha/\beta = -F'(0)/f''(0)$ further divides the surface into two regions; to the left of this line $\xi < \alpha/\beta$, the shear on the wall is to the right, while to the right, $\xi > \alpha/\beta$, this same shear is to the left. Thus the line $\xi = \alpha/\beta$ is a locus of surface shear lines dividing the flow into "upstream" and "downstream" regions and is analogous to a three-dimensional separation line. However, it does not have associated with it the other manifestations of "separation." We show schematically in Fig. 3 some of the features of this special case.

We have thus shown the three-dimensional analog of Rott's two-dimensional separation. The above solution for the three-dimensional stagnation point over a moving wall can be generalized to the compressible case, but we do not do so here.

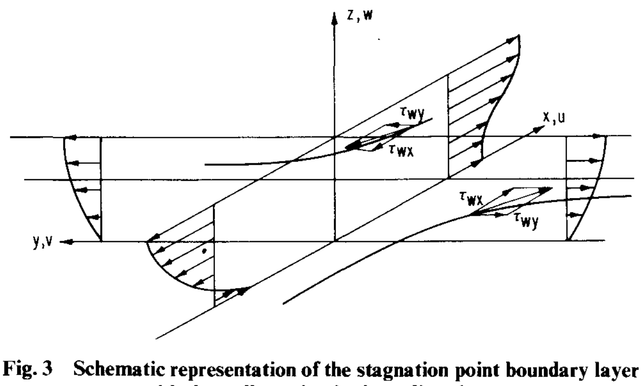

Fig. 3 Schematic representation of the stagnation point boundary layer with the wall moving in the x-direction.

‡ Because they do not appear to be relevant for present purposes we do not consider the "discontinuous branch" solutions for $c < 0$, cf. Ref. 6.

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Incompressible Potential Flow Solutions for Arbitrary Bodies of Revolution

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I. Introduction

The potential flow about an arbitrary body of revolution was first treated by Theodore von Kármán.[1] He determined the potential flow around bodies of revolution at zero angle of attack by superposing a uniform stream on a system of line sources distributed along the axis of the body. The strengths of the sources were determined so that the zero streamline passed through given coordinates of the body, the number of coordinates being equal to the number of sources.

Since the original work of von Kármán, very little enlightening

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information on this method has been published. While most fluid mechanics texts refer to this method, they generally do little more than recognize its existence and make no mention of its accuracy or limitations. An exception to this is the text by Karamcheti.[2] He states that there are certain conditions the body must meet before it can be exactly represented by a system of axial line sources; namely, that the body must be relatively slender and that there must be no discontinuities in the body's surface slope. The first condition is rather vague because "slender" is a matter of degree. The second condition is precise; however, in the practical application of potential flow theory, e.g., pressure distribution for a boundary-layer calculation, the matter of a discontinuity in the surface slope is usually not considered. The present Note deals with the accuracy and limitations of von Kármán's method of axial line sources for the calculation of incompressible potential flow about bodies of revolution at zero angle of attack.

II. Analysis

Consider the axisymmetric flow generated by N line sources aligned with a uniform stream. By means of elementary analysis it can be shown[3] that the Stokes stream function for such a flow is

$$\Psi(x,r) = U_\infty r^2/2-(1/4\pi)\sum_{j=1}^{N} Q_j[1+(D_j'-D_j'')/a] \qquad (1)$$

where U_∞ is the speed of the uniform stream, Q_j is the volume flow rate of the jth source, D_j' is the distance from the front of the jth source to the point of interest, D_j'' is the distance from the rear of the jth source to the point of interest, and a is the length of each line source.

In von Kármán's method a streamline, say the zero streamline, is forced to pass through a given set of body coordinates. In order to have a unique set of source strengths which can satisfy the specified coordinates of the zero streamline, the number of coordinates must be equal to the number of sources. Letting the coordinates of the zero streamline be (x_i, r_i) where $i = 1,2,\ldots N$, from Eq. (1) we have

$$0 = U_\infty r_i^2/2-(1/4\pi)\sum_{j=1}^{N} Q_j[1+(D_{i,j}'-D_{i,j}'')/a] \qquad i = 1,2,\ldots N \qquad (2)$$

where $D_{i,j}'$ is the distance from the front of the jth line source to the point (x_i, r_i), and similarly for $D_{i,j}''$.

It is seen from Eq. (2) that we have a linear system of algebraic equations of the form $[A]\{Q\} = \{C\}$, where $[A]$ is an $N\times N$ coefficient matrix, $\{Q\}$ is an $N\times 1$ vector of the unknown source strengths, and $\{C\}$ is an $N\times 1$ constant vector. When this system is solved for the vector $\{Q\}$, the stream function for the entire field can be calculated. From the stream function it is a simple matter to calculate the velocity components v_x, v_r and the surface pressure variation.

III. Discussion of Results

The first comparison between von Kármán's method and analytical results was made for flow around a sphere. The sphere flowfield was generated by 20 line sources distributed continuously from the front to the rear of the sphere. The zero streamline generated by von Kármán's method passed through all of the specified body coordinates. It was noted, however, that because of the symmetrical geometry of the sphere the line source strengths should display an antisymmetric character. That is, the values of the source strengths from the front to the center of the sphere should be the same as the values from the rear to the center of the sphere, except opposite in sign. The calculated source strengths did not display this character. It was conjectured that computer roundoff error in the matrix solution technique‡ was the cause of this. The entire solution technique was changed to double precision, which on the CDC 6600 is 29_{10} significant figures. The result of using double precision was a definite change in the solution; the new solution was antisymmetric to eight significant figures. The need for double precision in the calculation indicates that the system of equations for the source strengths is ill-conditioned. This caused us to use double precision in all subsequent calculations.

‡ Gauss elimination with partial pivoting was used.

The next topic investigated was the effect the number of sources had on generating the sphere. As a test case, a sphere was generated using 18 sources. The surface velocity was calculated for this body and it compared well with the surface velocity calculated using 20 sources, except near the stagnation points. By examining the source strengths of the 18 source and 20 source solutions it was found that the lead source of the 18 source solution was a positive source and the lead source of the 20 source solution was a negative source, i.e., a sink. The result is that there is no stagnation point for the zero streamline using 20 line sources because the zero streamline actually passes into the lead source. On the other hand, for a sphere generated by 18 sources, the stagnation point was slightly ahead of the lead source because the lead source had a positive efflux. The differences in the two cases are noticeable only very near the stagnation points, but a positive efflux for the first source, as in the case of 18 sources, is more desirable because it gives a realistic pressure distribution near the front stagnation point.

The calculation of potential flow around a sphere was also attempted using an odd number of sources instead of an even number as previously discussed. The results for an odd number of sources were very peculiar indeed. While the zero streamline did pass through all of the specified coordinates of the sphere, the body generated had "holes" in its surface. That is, between the specified coordinates the zero streamline plunged into the negative strength line sources on the axis and then reappeared so that it passed through the next specified coordinate. This behavior produced, as one would imagine, highly unrealistic velocities. It is hypothesized that this erratic behavior was caused by the fact that with an odd number of line sources one of the sources must overlap the center of the sphere and, consequently, prohibit any antisymmetric solution. Therefore, odd numbers of sources must be avoided for bodies that are symmtric about a plane normal to their axis at the midpoint of their length.

The next potential flowfield that was calculated was the flow around several Rankine ovals of varying fineness ratio. A sphere, of course, is the limiting case for Rankine ovals, so the fineness ratio was varied from 1 to 10. In general, the results of these calculations gave very good agreement with the analytical solution. Typical results are shown in Figs. 1 and 2 for fineness ratios of 4 and 10, respectively.

A slight rippling phenomenon was noticed in the surface of high fineness ratio ovals. This is due to the proximity of the line sources to the surface of high fineness ratio bodies. This proximity

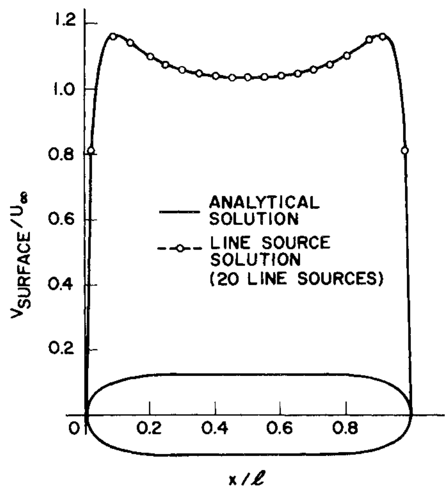

Fig. 1 Comparison of analytical and line source solutions for a Rankine oval of fineness ratio 4.

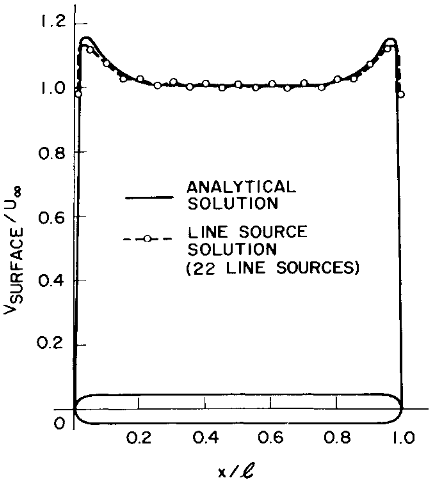

Fig. 2 Comparison of analytical and line source solutions for a Rankine oval of fineness ratio 10.

to the line sources causes an increased local effect of the line sources which, as a result, gives rise to the surface rippling. The rippling was more pronounced using 20 sources as compared to 40 sources because the surface was less restricted with the smaller number of sources. The surface velocity using 20 sources was still within 1% of analytical values, so the solution was considered adequate.

An attempt was made to duplicate the velocity distribution about a body given by Smith and Pierce.[4] The shape was a smooth body of revolution with a large annular bump near the nose (Fig. 3). Smith and Pierce state that the method of axial line sources predicts an incorrect velocity distribution over the body. A more accurate statement, however, of their conclusion is that the zero streamline for a system of line sources cannot follow the contour of the body, and therefore, cannot be used to calculate the velocities around it. The results for line sources given by Smith and Pierce were obtained by reducing the number of body coordinates until the zero streamline could pass thrugh all of them, but these were too few to accurately describe the body. This example showed that the method of axial line sources cannot generate a body with relatively large deviations in surface slope.

Another class of bodies that were generated by von Kármán's method was cones. The results for the potential flow around cones were generally good when the included angle of the cone was less than a certain value. The only part of the cone that was not accurately generated, as would be expected, was the cone tip. The tip of the generated cone was slightly rounded because the lead line source was positive. It was found from numerical experiments that for a given number of sources there is a maximum included angle the cone may have before the method breaks down. For 20 sources, the maximum included angle was found to be 51.8°. If it is attempted to generate a cone with this included angle with 40 sources, the solution will again break down. It is evident, therefore, that the method of axial line source is not only sensitive to the contour of the body, but also the number of sources used to generate it.

IV. Conclusions

Von Kármán's method of axial line sources produces a linear system of simultaneous equations which, in general, is ill-conditioned. Very high accuracy is, therefore, required in calculating the equations and finding their solution. Our results indicate that the method of axial line sources is not generally practical on digital computers that have less than 20_{10} or 25_{10} significant figure accuracy.

Von Kármán's method does not always produce reliable solutions for the potential flow around a specified body. To determine if the solution is accurate, at least two checks must be made; first, check if the zero streamline passes through all the specified coordinates of the body, and second, check if the velocity is well behaved along the body surface.

To determine if von Kármán's method is applicable for a certain body, it is not sufficient to state simply that the body need only be slender and lacking discontinuities in surface slope. The present investigation concludes that there are several interrelated factors, some of which were documented here, that determine if an accurate solution can be obtained.

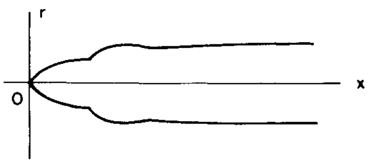

Fig. 3 Body of revolution with large annular bump as given by Ref. 4.

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Shock Shapes of Blunt Bodies in Hypersonic Helium, Air, and CO_2 Flows

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Nomenclature

M_∞ = freestream Mach number
$N_{Re,\infty}$ = freestream unit Reynolds number, m^{-1}
p_t = stagnation point pressure, kN/m^2
p_∞ = freestream static pressure, kN/m^2
r_b = model base radius, m
r_n = spherical nose radius, m
T_t = stagnation point temperature, °K
T_∞ = freestream temperature, °K
U_∞ = freestream velocity, km/sec
x, r = cylindrical coordinates (see Fig. 2)
γ = ratio of specific heats
γ_{eff} = effective ratio of specific heats
$\gamma_{E,2}$ = postnormal shock static isentropic exponent
$\gamma_{E,\infty}$ = freestream isentropic exponent
Δ = ratio of shock standoff distance to model base radius
ε = normal-shock density ratio
ρ_∞ = freestream density, gm/m^3
θ = cone semiapex angle, deg

Introduction

A PRIMARY factor governing hypersonic flowfield characteristics of blunt vehicles entering planetary atmospheres is

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